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# Distortion theorems for convex mappings on homogeneous balls

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ABSTRACT

Let $B$ be the open unit ball of a complex Banach space $X$ and let $B$ be homogeneous. We prove distortion results for normalized convex mappings $f : B \to X$ which generalize various finite dimensional distortion theorems and improve some infinite dimensional ones. In particular, our results are valid for the open unit balls of complex Hilbert spaces and the Cartan domains.



## 1. Introduction

The object of this paper is to generalize the distortion theorems for convex mappings on finite dimensional Euclidean balls to infinite dimension, as well as improving some infinite dimensional results. From the perspective of the Riemann Mapping Theorem, an appropriate generalization of the open unit disc in the complex plane $\mathbb{C}$ would be the open unit ball $B$ of a complex Banach space such that $B$ is homogeneous. Indeed, it has been shown in [14] that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to such a ball. A complex Banach space is a JB*-triple if, and only if, its open unit ball is homogeneous. Therefore a natural extension of the finite dimensional distortion theorems should be the ones for convex mappings on the open unit ball of a JB*-triple. We achieve such an extension by proving the following distortion results.

**Theorem 1.1.** *Let $B$ be the open unit ball of a JB*-triple $X$. Given a normalized convex mapping $f : B \to X$ with derivative $Df$, we have, for $a, b, x \in B$ and $y \in X$,*

(i) $\frac{1}{(1+\|x\|)^2} \leqslant \|Df(x)\| \leqslant \frac{1}{(1-\|x\|)^2}$;
(ii) $\frac{(1-\|x\|)\|y\|}{(1+\|x\|)\|B(x,x)^{1/2}\|} \leqslant \|Df(x)y\| \leqslant \frac{\|y\|}{(1-\|x\|)^2}$;
(iii) $\|f(a) - f(b)\| \geqslant \frac{\sinh C_B(a,b)}{\exp C_B(a,b)} \max\{\frac{1-\|a\|^2}{\|[Df(a)]^{-1}\|}, \frac{1-\|b\|^2}{\|[Df(b)]^{-1}\|}\}$;
(iv) $\|f(a) - f(b)\| \leqslant \sinh C_B(a,b) \exp C_B(a,b) \min\{\|B(a,a)^{1/2}\|\|Df(a)\|, \|B(b,b)^{1/2}\|\|Df(b)\|\}$,

*where $C_B$ is the Carathéodory distance on $B$, $B(x,x) : X \to X$ is the Bergmann operator and $\|B(x,x)^{1/2}\| \leqslant 1$.*

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We prove the above theorem in a series of lemmas and estimate the operator norm $\|B(x,x)^{1/2}\|$ in terms of $\|x\|$ or other values intrinsic to $x$. For convenience, we will refer to the open unit ball of a JB*-triple as a *homogeneous ball*. Homogeneous balls include the classical Cartan domains (cf. [13]) as well as the two exceptional domains, the open unit balls of complex Hilbert spaces and the open unit balls of C*-algebras. A key to our results is some estimates involving the Möbius transformation and the Bergmann operator on a homogeneous ball which may be of some independent interest.

We refer to [5,7] for reference and motivation for distortion results in higher dimensions. For the open unit disc $U$ in $\mathbb{C}$, the distortion theorem in Theorem 1.1(i) for convex functions on $U$ is well known. This result has been extended to the Euclidean balls in $\mathbb{C}^n$ by Gong and Liu [6], Pfaltzgraff and Suffridge [19]. It has been further generalized to the open unit balls of complex Hilbert spaces by Hamada and Kohr [10,11]. In [10], it is proved that the upper bound in Theorem 1.1(i) is sharp for the open unit balls of complex Hilbert spaces although the lower bound is not sharp for the Euclidean balls of dimension at least 2 (cf. [18]). Zhu and Liu [21] have obtained the following distortion theorem for convex mappings $f$ on the open unit balls of complex Banach spaces:

$$\frac{1}{(1+\|x\|)^2} \leqslant \|Df(x)\| \leqslant \frac{1+\|x\|}{(1-\|x\|)^2}.$$

They [21, Conjecture 2.2] conjectured that Theorem 1.1(i) holds for convex mappings on the open unit balls of complex Banach spaces. Our theorem is an affirmative answer to this conjecture for convex mappings on homogeneous balls.

It has been proved in [21] the following distortion theorem for convex mappings on the open unit balls of complex Hilbert spaces:

$$\frac{\|y\|\sqrt{1-\|x\|^2}}{(1+\|x\|)^2} \leqslant \|Df(x)y\| \leqslant \frac{\|y\|}{(1-\|x\|)^2}. \tag{1.1}$$

This result is generalized in Theorem 1.1(ii) to homogeneous balls. In fact, we show that $\|B(x,x)^{1/2}\| = \sqrt{1-\|x\|^2}$ for complex Hilbert spaces of dimension greater than 1.

The two-point distortion in Theorem 1.1(iii) for the open unit disc in $\mathbb{C}$ has been proved in [16]. On finite dimensional Euclidean balls, it has been proved by Kohr [17] (cf. [5]) for convex mappings and it has also been proved by Graham, Kohr and Pfaltzgraff [8] for a linear invariant family of biholomorphic mappings (of particular finite norm order) (cf. [4,7]). In fact, both (iii) and (iv) in Theorem 1.1 also hold for nonnormalized convex mappings. On the other hand, a locally biholomorphic mapping $f: B \to X$ satisfying (iii) must be biholomorphic.

## 2. Distortion theorems

Throughout all Banach spaces are complex. Let $X$ and $Y$ be Banach spaces with domains $G \subset X$ and $D \subset Y$. We denote by $H(G,D)$ the set of holomorphic mappings from $G$ into $D$. For $f \in H(G,D)$ and $x \in G$, let $Df(x)$ be the Fréchet derivative of $f$ at $x$. A mapping $f \in H(G,X)$ is called *normalized* if $f(0)=0$ and the derivative $Df(0): X \to X$ is the identity map. A mapping $f \in H(G,Y)$ is said to be *biholomorphic* if $f(G)$ is a domain, the inverse $f^{-1}$ exists on $f(G)$ and is holomorphic on $f(G)$. A biholomorphic mapping $f \in H(G,Y)$ is called *convex* if the image $f(G)$ is convex in $Y$. Let $\mathrm{Aut}(G)$ be the automorphism group of biholomorphic mappings of $G$ onto itself. A domain $G$ is called *homogeneous* if for any $x, y \in G$, there exists an $f \in \mathrm{Aut}(G)$ such that $f(x)=y$. Every bounded symmetric domain in a complex Banach space is homogeneous. Conversely, the open unit ball $B$ of a Banach space admits a symmetry $s(x)=-x$ at 0 and if $B$ is homogeneous, then $B$ is a symmetric domain. Banach spaces with a homogeneous open unit ball are precisely the JB*-triples [14]. They are the complex Banach spaces $X$ equipped with a triple product $\{\cdot,\cdot,\cdot\}: X^3 \to X$ which is conjugate linear in the middle variable, but linear and symmetric in the other variables, and satisfies

(i) $\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}$;
(ii) the map $a\Box a: x \in X \mapsto \{a,a,x\} \in X$ is hermitian with nonnegative spectrum;
(iii) $\|\{a,a,a\}\| = \|a\|^3$

for $a,b,x,y,z \in X$. A closed subspace of a JB*-triple is called a *subtriple* if it is closed with respect to the triple product. A JB*-triple is called a *JC*-triple* or *J*-algebra* if it can be embedded as a subtriple of the JB*-triple $L(H)$ of bounded linear operators on a Hilbert space $H$, with the usual triple product

$$\{x,y,z\} = \frac{1}{2}\big(xy^*z + zy^*x\big) \quad \big(x,y,z \in L(H)\big),$$

where $y^*$ denotes the adjoint of $y$. We refer to [2,20] for relevant details of JB*-triples and references.

Besides the box operator $a\Box a$, a fundamental operator on a JB*-triple $X$ is the Bergmann operator $B(a,b): X \to X$ defined by

$$B(a,b)(x) = x - 2\{a,b,x\} + \big\{a,\{b,x,b\},a\big\} \quad (x \in X).$$

On the open unit ball $B$ of a JB*-triple, each point $a \in B$ induces the Möbius transformation $g_a \in \mathrm{Aut}(B)$ given by

$$g_a(x) = a + B(a,a)^{1/2}(I + x\Box a)^{-1}x \quad (x \in X),$$

where $x\Box a$ is the box operator $(x\Box a)(y) = \{x, a, y\}$. We have $g_a(0) = a$, $g_a^{-1} = g_{-a}$ and

$$Dg_a(0) = B(a,a)^{1/2}, \qquad Dg_{-a}(a) = B(a,a)^{-1/2}.$$

Sharp estimates for distortion theorems depend on precise values of the norms of positive and negative square roots of the Bergmann operator $B(a,a)$. Indeed, on the one-dimensional JB*-triple $\mathbb{C}$, we have the exact values from $B(a,a)^{1/2}(z) = (1-|a|^2)z$ for $z \in \mathbb{C}$. In any dimension, we have

$$\left\| B(a,a)^{-1/2} \right\| = \frac{1}{1-\|a\|^2} \tag{2.1}$$

from [15, Corollary 3.6]. We remark that each $g \in \mathrm{Aut}(B)$ is a composite of a Möbius transformation and a linear isometry, by Cartan's uniqueness theorem, hence it is true that

$$\left\| \left[ Dg(0) \right]^{-1} \right\| = \frac{1}{1-\|a\|^2} \tag{2.2}$$

whenever $g \in \mathrm{Aut}(B)$ satisfies $g(0) = a$ (see also [9]).

Although $1 - \|a\|^2 \leqslant \|B(a,a)^{1/2}\| \leqslant 1$, the upper bound 1 here can be improved to one in terms of numerical values associated to $a$, for JC*-triples.

If $X$ is a JC*-triple contained in some $L(H)$, then the Bergmann operator has the form

$$B(a,a)^{1/2}(x) = \left(\mathbf{1} - aa^*\right)^{1/2} x \left(\mathbf{1} - a^*a\right)^{1/2} \quad \left(x \in L(H)\right)$$

(cf. [12, Theorem 2]), where $\mathbf{1}$ denotes the identity operator in $L(H)$. Hence we have

$$\begin{aligned} \left\| B(a,a)^{1/2} \right\| &\leqslant \left\| \left(\mathbf{1} - aa^*\right)^{1/2} \right\| \left\| \left(\mathbf{1} - a^*a\right)^{1/2} \right\| \\ &= \left\| \mathbf{1} - aa^* \right\|^{1/2} \left\| \mathbf{1} - a^*a \right\|^{1/2} \\ &\leqslant 1. \end{aligned} \tag{2.3}$$

One can compute the norm $\|\mathbf{1} - aa^*\|$ via the Hilbert space $H$. For each $a \in X \subset L(H)$, we define

$$\alpha(a) = \inf\left\{ \|a\xi\| \colon \xi \in H,\ \|\xi\| = 1 \right\}$$

and

$$\beta(a) = \inf\left\{ \|a^*\xi\| \colon \xi \in H,\ \|\xi\| = 1 \right\}.$$

**Lemma 2.1.** *Let $B$ be the open unit ball of a JC*-triple $X \subset L(H)$ and let $a \in B$. We have $\|B(a,a)^{1/2}\|^2 \leqslant (1 - \alpha(a)^2)(1 - \beta(a)^2)$.*

**Proof.** This follows from (2.3) and $\|(\mathbf{1} - aa^*)^{1/2}\|^2 = \sup_{\|\xi\|=1} \langle (\mathbf{1} - aa^*)\xi, \xi \rangle = 1 - \beta(a)^2$ as well as $\|(\mathbf{1} - a^*a)^{1/2}\|^2 = 1 - \alpha(a)^2$. □

**Remark 2.2.** In the above lemma, if X is a $C^*$-algebra with identity, then one can obtain an estimate of the upper bound independent of the underlying complex Hilbert space $H$:

$$\left\| B(a,a)^{1/2} \right\|^2 \leqslant \left(1 - \inf \sigma\left(aa^*\right)\right)\left(1 - \inf \sigma\left(a^*a\right)\right),$$

where $\sigma(aa^*)$ and $\sigma(a^*a)$ denote respectively the spectrum of $aa^*$ and of $a^*a$.

Indeed, this can be seen easily from functional calculus, since the C*-subalgebra of $X$ generated by $aa^*$ and the identity $\mathbf{1}$ is C*-isomorphic to the abelian C*-algebra $C(\sigma(aa^*))$ of complex continuous functions on the spectrum $\sigma(aa^*) \subset [0, \infty)$, with $aa^*$ identifies with the nonnegative function $id : \sigma(aa^*) \to \sigma(aa^*)$. Hence $\|\mathbf{1} - aa^*\| = 1 - \inf \sigma(aa^*)$ and likewise $\|\mathbf{1} - a^*a\| = 1 - \inf \sigma(a^*a)$.

We now consider the open unit ball of a complex Hilbert space. A complex Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ is a JB*-triple in the triple product

$$\{x, y, z\} = \frac{1}{2}\left(\langle x, y\rangle z + \langle z, y\rangle x\right)$$

and carries the structure of a JH-triple (cf. [1]). The Bergmann operator $B(a,a) \colon H \to H$ is a positive operator for $\|a\| < 1$.

**Lemma 2.3.** *Let $B$ be the open unit ball of a complex Hilbert space $H$ and let $a \in B$. Then we have*

$$\left\|B(a,a)^{1/2}\right\|^2 = \left\|B(a,a)\right\| = \begin{cases} (1-\|a\|^2)^2 & \text{if } \dim H = 1, \\ 1-\|a\|^2 & \text{if } \dim H \geqslant 2. \end{cases}$$

**Proof.** For each $z \in H$, we have

$$B(a,a)(z) = z - 2\{a,a,z\} + \big\{a,\{a,z,a\},a\big\} = \big(1-\|a\|^2\big)\big(z - \langle z,a\rangle a\big),$$

where

$$\begin{aligned}\left\|z-\langle z,a\rangle a\right\|^2 &= \big\langle z-\langle z,a\rangle a, z-\langle z,a\rangle a\big\rangle \\ &= \|z\|^2 - 2\big|\langle z,a\rangle\big|^2 + \big|\langle z,a\rangle\big|^2\|a\|^2 \\ &< \|z\|^2 - \big|\langle z,a\rangle\big|^2 \leqslant \|z\|^2.\end{aligned}$$

It follows that $\|B(a,a)(z)\| \leqslant (1-\|a\|^2)\|z\|$.

If $H = \mathbb{C}$, we have actually the equality $|B(a,a)(z)| = (1-|a|^2)^2|z|$.

If $\dim H \geqslant 2$, we can pick a unit vector $z_0 \in H$ orthogonal to $a$ and, therefore,

$$\left\|B(a,a)\right\| \geqslant \left\|B(a,a)z_0\right\| = \left\|\big(1-\|a\|^2\big)z_0\right\| = 1-\|a\|^2$$

which proves that $\|B(a,a)\| = 1-\|a\|^2$. □

Let $U$ be the open unit disc in $\mathbb{C}$ and $G$ a domain in a Banach space $X$. For each $(x,\xi) \in G \times X$, the infinitesimal Carathéodory pseudometric $\gamma_G(x,\xi)$ on $G$ is defined by

$$\gamma_G(x,\xi) = \sup\big\{\big|Dh(x)\xi\big|\colon h \in H(G,U),\ h(x) = 0\big\}.$$

Each $\varphi \in \operatorname{Aut}(G)$ is an isometry in this pseudometric:

$$\gamma_G(x,\xi) = \gamma_G\big(\varphi(x), D\varphi(x)\xi\big)$$

and for the open unit ball $B$ in a Banach space $X$, one has $\gamma_B(0,\xi) = \|\xi\|$. We note that $\gamma_U$ is the Poincaré metric on $U$ (cf. [3, p. 54]).

We will make use of the following distortion theorem in [11, Remark 4] (see also [21, Theorem 2.1]).

**Lemma 2.4.** *Let $B$ be the open unit ball in a Banach space $X$ and let $f: B \to X$ be a normalized convex mapping on $B$. Then we have*

$$\frac{1-\|x\|}{1+\|x\|}\gamma_B(x,y) \leqslant \left\|Df(x)y\right\| \leqslant \frac{1+\|x\|}{1-\|x\|}\gamma_B(x,y) \tag{2.4}$$

*for each $x \in B$ and $y \in X$.*

Now we are ready to show Theorem 1.1(i). The lower bound there has already been obtained in [21]. The following lemma will establish the upper bound in Theorem 1.1(i) as well as the right inequality in Theorem 1.1(ii).

**Lemma 2.5.** *Let $B$ be the open unit ball of a JB*-triple $X$ and let $f: B \to X$ be a normalized convex mapping. Then we have*

$$\left\|Df(x)y\right\| \leqslant \frac{\|y\|}{(1-\|x\|)^2} \quad (x \in B,\ y \in X).$$

**Proof.** Let $x \in B$. Then the Möbius transformation $g_{-x} \in \operatorname{Aut}(B)$ satisfies $g_{-x}(x) = 0$. Let $y \in X$. From (2.1) and (2.4), we deduce that

$$\begin{aligned}\left\|Df(x)y\right\| &\leqslant \frac{1+\|x\|}{1-\|x\|}\gamma_B(x,y) = \frac{1+\|x\|}{1-\|x\|}\gamma_B\big(0, Dg_{-x}(x)y\big) \\ &= \frac{1+\|x\|}{1-\|x\|}\left\|B(x,x)^{-1/2}y\right\| \\ &\leqslant \frac{\|y\|}{(1-\|x\|)^2}. \qquad \square\end{aligned}$$

We prove likewise the left inequality in Theorem 1.1(ii).

**Lemma 2.6.** *Let $B$ be the unit ball of a JB*-triple $X$ and let $f : B \to X$ be a normalized convex mapping. Then for $(x, y) \in B \times X$, we have*

$$\frac{(1-\|x\|)\|y\|}{(1+\|x\|)\|B(x,x)^{1/2}\|} \leqslant \|Df(x)y\|.$$

**Proof.** Let $x \in B$ and $y \in X$. Applying the Möbius transformation $g_{-x} \in \mathrm{Aut}(B)$ and (2.4) again, we get

$$\begin{aligned}\|Df(x)y\| &\geqslant \frac{1-\|x\|}{1+\|x\|}\gamma_B(x,y)\\ &= \frac{1-\|x\|}{1+\|x\|}\gamma_B\big(0, Dg_{-x}(x)(y)\big)\\ &= \frac{1-\|x\|}{1+\|x\|}\|Dg_{-x}(x)(y)\|,\end{aligned}$$

where $\|y\| = \|B(x,x)^{1/2}Dg_{-x}(x)(y)\| \leqslant \|B(x,x)^{1/2}\|\|Dg_{-x}(x)(y)\|$ gives

$$\|Df(x)y\| \geqslant \frac{(1-\|x\|)\|y\|}{(1+\|x\|)\|B(x,x)^{1/2}\|}. \qquad \square$$

If $B$ is the open unit ball of a JC*-triple, we observe from the above and Lemma 2.1 that

$$\frac{(1-\|x\|)\|y\|}{(1+\|x\|)\sqrt{1-\alpha(x)^2}\sqrt{1-\beta(x)^2}} \leqslant \|Df(x)y\|$$

and if $B$ is the open unit ball of a complex Hilbert space, Lemma 2.3 implies that

$$\frac{(1-\|x\|)\|y\|}{(1+\|x\|)\sqrt{1-\|x\|^2}} \leqslant \|Df(x)y\|$$

which is identical to the lower estimate in (1.1) (cf. [10] and [11, Theorem 8]).

Finally, we prove the two-point distortions in Theorem 1.1(iii) and (iv). Given a normalized convex mapping $f : B \to X$ on the open unit ball $B$ of a Banach space $X$, we have the following growth theorem from [10, Theorem 2.1]:

$$\frac{\|x\|}{1+\|x\|} \leqslant \|f(x)\| \leqslant \frac{\|x\|}{1-\|x\|}. \tag{2.5}$$

For $a, b \in B$, the Carathéodory distance $C_B(a, b)$ is defined by

$$C_B(a,b) = \sup\big\{\rho\big(g(a), g(b)\big):\ g \in H(B,U)\big\}$$

where $\rho$ is the Poincaré distance on the open unit disc $U$ and we have, for $z \in B$,

$$C_B(z,0) = \frac{1}{2}\log\left(\frac{1+\|z\|}{1-\|z\|}\right).$$

Note that $C_B$ is invariant under the automorphism group $\mathrm{Aut}(B)$.

Using the estimates (2.5) and arguments similar to those in the proof of [8, Theorem 7], we obtain the following result.

**Lemma 2.7.** *Let $B$ be a homogeneous ball and let $f : B \to X$ be a convex mapping. Then for $a, b \in B$, we have*

$$\|f(a)-f(b)\| \geqslant \frac{\sinh C_B(a,b)}{\exp C_B(a,b)} \max\big\{\big\|\big[Df(a)D\phi_a(0)\big]^{-1}\big\|^{-1}, \big\|\big[Df(b)D\phi_b(0)\big]^{-1}\big\|^{-1}\big\}$$

*and*

$$\|f(a)-f(b)\| \leqslant \sinh C_B(a,b)\exp C_B(a,b)\min\big\{\|Df(a)D\phi_a(0)\|, \|Df(b)D\phi_b(0)\|\big\},$$

*where $\phi_a, \phi_b \in \mathrm{Aut}(B)$ satisfy $\phi_a(0) = a$ and $\phi_b(0) = b$.*

**Proof.** The first estimate can be obtained by the same arguments as in the proof of [8, Theorem 7]. To prove the upper bound for $\|f(a) - f(b)\|$, fix $a, b \in B$ and define $F : B \to X$ by

$$F(x) = \big[D\phi_a(0)\big]^{-1}\big[Df(a)\big]^{-1}\big(f\big(\phi_a(x)\big) - f(a)\big) \quad (x \in B).$$

Then $F$ is a normalized convex mapping on $B$. In view of (2.5), we have

$$\|F(x)\| \leqslant \frac{\|x\|}{1-\|x\|} = \sinh C_B(x,0)\exp C_B(x,0)$$

for all $x \in B$. It follows that, for $x = \phi_a^{-1}(b)$,

$$\begin{aligned}\|f(b)-f(a)\| &= \|Df(a)D\phi_a(0)F(x)\| \leqslant \|Df(a)D\phi_a(0)\| \cdot \|F(x)\| \\ &\leqslant \|Df(a)D\phi_a(0)\| \cdot \sinh C_B(x,0)\exp C_B(x,0) \\ &= \|Df(a)D\phi_a(0)\| \cdot \sinh C_B(a,b)\exp C_B(a,b).\end{aligned}$$

Changing the roles of $a$ and $b$, one deduces the desired result. $\square$

In the above lemma, we have, by (2.2),

$$\|[Df(a)D\phi_a(0)]^{-1}\| \leqslant \frac{\|[Df(a)]^{-1}\|}{1-\|a\|^2}$$

and

$$\|[Df(b)D\phi_b(0)]^{-1}\| \leqslant \frac{\|[Df(b)]^{-1}\|}{1-\|b\|^2}$$

which yields readily Theorem 1.1(iii). We remark that the two-point distortion theorem in [16, Remark] for the open unit disc $U$ in $\mathbb{C}$ is a special case of the above result.

Also, since

$$\|Df(a)D\phi_a(0)\| \leqslant \|Df(a)\|\|B(a,a)^{1/2}\|$$

and

$$\|Df(b)D\phi_b(0)\| \leqslant \|Df(b)\|\|B(b,b)^{1/2}\|,$$

we obtain Theorem 1.1(iv), where $\|B(x,x)^{1/2}\| = \sqrt{1-\|x\|^2}$ for $x = a, b$ in the open unit ball of a complex Hilbert space of dimension at least 2.

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